

Momentum Force and Energetic Force:

A New Classification in Classical Mechanics

Newton's Second Law:

$$m\mathbf{a} = \mathbf{F}_{momentumic} + \mathbf{F}_{energetic}$$

Alternative Lagrange's Equations of Motion:

$$J_P(\dot{q})\ddot{q} = \underbrace{\left(J_P^T(q) - J_P(q)\right)\dot{q}}_{generalized\ momentumic\ force} + \underbrace{Q - \frac{\partial h}{\partial q} - \frac{\partial^2 L}{\partial \dot{q} \partial t}}_{generalized\ energetic\ force}$$

A Theoretical Note by Saeed Montazeri

Momentum Force and Energetic Force: A New Classification in Classical Mechanics

Saeed Montazeri

2025

© 2025 Saeed Montazeri

This work is distributed under the Creative Commons Attribution (CC BY 4.0) license.

This booklet is an expanded version of a preprint previously released by the author on ResearchGate (DOI: <http://dx.doi.org/10.13140/RG.2.2.30725.56800>)

Author's information

Email: saeed.montazeri@outlook.com

ORCID ID: <https://orcid.org/0000-0002-4780-2014>

Table of Contents

1. Introduction	1
1.1 Historical Overview of Force	1
1.2 Types of Force Classification	2
1.3 Aims of the Booklet	2
2. New Definitions	4
Definition 1: Momentum Force	4
Definition 2: Energetic Force	4
Definition 3: Momentum-Energetic Form	4
Theorem 1 and Theorem 2	5
3. Existence of Momentum Force in Nature	7
3.1 Magnetic Force	7
3.2 Constraint Force	7
3.3 Magnus Force	7
4. Applications of Momentum-Energetic Form	8
Case 1: Eulerian Specification	8
Case 2: The Momentum-Energetic Form in Generalized Space: Reformulating Lagrange's Equations of Motion.....	9
Case 3: The Momentum-Energetic Form for Conservative Velocity Dependent Forces.....	15
5. Discussion	16
6. Summary	16
References	17

Preface

The purpose of this booklet is not to discover new forces in mechanical physics, but to present a new qualitative and conceptual perspective on forces. This booklet suggests that forces be classified based on their ability to do mechanical work. There exists a class of forces in nature that, since they are always perpendicular to the velocity, are incapable of doing mechanical work and can only influence the change of momentum. For example, the magnetic force, forces caused by time-independent holonomic constraints, and the Magnus force all belong to this type.

Prior to the presentation of this booklet, no one had proposed any naming or classification for this group of forces. This booklet suggests that this category of forces be named “momentumic forces.” On the other hand, forces capable of performing mechanical work are introduced in this booklet as “energetic forces.” The usefulness of such a classification becomes evident when formulating the energy balance equation. In general, if two different forces, $\mathbf{F}_a$ and $\mathbf{F}_b$, do the same amount of work, then a third force that relates these two forces to each other is a momentumic force. Finally, three cases of applications of this type of force classification are exemplified. In three-dimensional Euclidean space, it can be shown that any force $\mathbf{F}$ can be expressed as the resultant of two forces: energetic and momentumic. It is obvious that Newton's second law for a particle of mass m can be written in the form $m\mathbf{a} = \mathbf{F}_{\text{momentumic}} + \mathbf{F}_{\text{energetic}}$. The momentumic force can also be defined in generalized space. An interesting point proven in this booklet is that if forces are classified from this perspective, the Lagrange equations can also be exactly reformulated in the form $[\mathbf{M}][\mathbf{a}] = [\text{generalized momentumic force}] + [\text{generalized energetic force}]$. Moreover, it is demonstrated that the form of the equation for any velocity-dependent conservative force (such as the electromagnetic force) can be defined using the new concepts introduced in this booklet.

Historically, this booklet bears a remarkable resemblance to the paper in which Gaspard-Gustave de Coriolis introduced the concept of the compound centrifugal force, now known as the Coriolis force. The Coriolis force was already known before Coriolis's publication, and even Euler had derived its mathematical formula. The key contribution made by Coriolis was the introduction of a new name and terminology for this force. A similar kind of naming process is almost repeated in this work. The momentumic forces had already been recognized in nature, and their formulas were known, but there had been no uniform terminology for them. At the end of the booklet, by presenting a historical comparison of the Coriolis force and this work, and by citing relevant historical references, the author discusses why this booklet can introduce a new idea to classical mechanics.

1 Introduction

1.1 Historical Overview of Force

Aristotle may be regarded as the earliest philosopher to articulate a systematic account of the concept of force. His treatment of this notion was entirely qualitative and philosophical, relying on logical reasoning rather than quantitative analysis. In his framework, all local motion was classified into two principal categories: natural and violent. Natural motion encompassed both the uniform and eternal circular motion of celestial bodies and the rectilinear, upward or downward, and finite motion of terrestrial objects. Any motion that did not conform to these patterns was identified as violent motion, arising from an external cause [1]. Aristotle identified two fundamental notions of force. The first corresponds to the Platonic view, describing an internal principle of activity inherent in matter itself, which he called nature or “physis.” The second refers to an external cause that originates from a substance and is expressed through the actions of pushing and pulling. This kind of force produces motion in another body rather than in itself [2, pp. 35–36]. Aristotle writes:

“Nature is a cause of movement in the thing itself, force a cause in something else, or in the thing itself regarded as something else. All movement is either natural or enforced, and force accelerates natural motion (e.g., that of a stone downwards), and is the sole cause of unnatural” [3].

The Renaissance gradually questioned these long-held beliefs, as intellectuals started to scrutinize Aristotle’s classifications, opening the path toward a more mathematical approach to understanding nature. Galileo contributed significantly to this shift through his experiments and analyses, which clearly separated the description of motion (kinematics) from its causes (dynamics). His work laid the foundation for a concept of force that could be quantified [2, Ch. 6]. The idea appears in *Dialogues Concerning Two New Sciences* and is delivered by the character Salviati:

“The present does not seem to be the proper time to investigate the cause of the acceleration of natural motion concerning which various opinions have been expressed by various philosophers, some explaining it by attraction to the center, others to repulsion between the very small parts of the body, while still others attribute it to a certain stress in the surrounding medium which closes in behind the falling body and drives it from one of its positions to another. Now, all these fantasies, and others too, ought to be examined; but it is not really worth while. At present it is the purpose of our Author merely to investigate and to demonstrate some of the properties of accelerated motion (whatever the cause of this acceleration may be)” [4].

Newton’s *Philosophiae Naturalis Principia Mathematica (Principia)*, published in 1687, transformed natural philosophy by moving it away from earlier Aristotelian and Cartesian frameworks. He introduced an operational notion of force, something that can be quantified and expressed through mathematics and supported by experimental evidence. In doing so, Newton laid the groundwork for what became classical mechanics [5].

1.2 Types of Force Classification

Forces can be classified in various ways. From the viewpoint of how they are applied, forces can be classified into two major groups: contact forces and volumic forces [6, Ch. 5]. Forces applied by friction, tension, or springs are examples of forces classified as contact forces. On the other hand, gravitational force and electromagnetic force are categorized as volumic forces.

Forces, from the perspective of the source from which they are applied to the particles of a mechanical system, can be classified into two groups: internal and external. The force that the particles of a system exert on each other is called an internal force, and the force applied to the particles from outside the system is called an external force.

Forces can be classified into two groups based on the work they do on particles: conservative and non-conservative forces. If the work done by a force on a particle is independent of the particle's path and depends only on the initial and final positions of the particle, that force is called conservative. Otherwise, it is a non-conservative force.

A force, as a vector quantity, must be measured within a coordinate system. The force measured in an accelerating coordinate system is called inertial/fictitious force. Centrifugal force and Coriolis force are two famous examples of this type of force [6, Ch. 6].

Forces can be categorized into two types from the viewpoint of acceleration generation: static forces such as tensile and compressive forces in the members of a truss, and dynamic forces such as the propulsive force of an aircraft.

From a mathematical point of view, force can be represented in various forms such as a scalar, a vector, a tensor, or a generalized force. The primary form in which force is defined in Newtonian mechanics is the vector form. Accordingly, if a dynamical problem involves only one spatial dimension, the scalar form of force can be conveniently used. In the field of continuum mechanics, the tensor form is used to understand the distribution of internal forces within a continuous material [7, 8]. The generalized form of force is used in Lagrangian and Hamiltonian mechanics. One of the advantages of using the generalized force form in the analysis of mechanical systems is that constraint and redundant forces, which do not perform mechanical work, are eliminated, thereby simplifying the analysis of the problem [9].

1.3 Aims of the Booklet

This booklet suggests that forces be categorized based on their ability to do work. A simple reason that clarifies the need for defining such a new classification can be found in the energy balance equation. To provide greater clarity for readers, a brief example is presented below to illustrate why defining the momentum force can be useful. This simple example will be explained in more detail in Section (4).

Let the velocity vector of a particle be expressed in terms of its Cartesian position coordinates as $\mathbf{v} = \mathbf{v}(v_x(x, y, z), v_y(x, y, z), v_z(x, y, z))$. This is the Eulerian form of velocity, which is widely used in fluid mechanics. Obviously, the instantaneous power of the resultant force $\mathbf{F}$ acting on the particle is given by:

$$\mathbf{F} \cdot \mathbf{v} = \nabla k \cdot \mathbf{v} \quad (I)$$

Where k is the kinetic energy of the particle. From Equation (I), it can be concluded that the only mathematical relationship expressing the connection between $\mathbf{F}$ and ∇k can be written as follows:

$$\mathbf{F} = \nabla k + \mathbf{F}_m \quad (\text{II})$$

Here, $\mathbf{F}_m$ denotes the force that must be perpendicular to the velocity $\mathbf{v}$ for Equation (I) to hold. Suppose one encounters an equation similar to Equation (II) in a mechanical problem. The question arises as to what name has been assigned to $\mathbf{F}_m$ in the standard references of mechanical physics?

So far, no formal definition has been provided for the force $\mathbf{F}_m$. In this booklet, the author precisely fills this gap and names this force “momentumic force.” Furthermore, the existence of examples of this type of force in nature, such as the magnetic force, the constraint force, and the Magnus force, makes it necessary to provide a formal definition for this category of forces. Finally, three cases related to applications of such classification are presented.

2 New Definitions

The content presented in this booklet revolves around three new definitions proposed by the author. In the following, the three new definitions are presented at the outset.

Assumption:

In the following three definitions, it is assumed that the kinematic quantities of particles are measured relative to universal coordinates.

Definition 1:

Any force applied to a moving point mass with velocity $\mathbf{v}$, such that its mechanical power is continuously zero, is called a **momentumic force**. The momentumic force can be expressed as follows:

$$\mathbf{F} = \mathbf{G} \times \mathbf{v} \quad (1)$$

Where $\mathbf{G}$ is called a **generator vector**. Every momentumic force has its corresponding generator vector.

Definition 2:

Any force applied to a moving point mass with velocity $\mathbf{v}$, such that its mechanical power is not continuously zero, is called an **energetic force**.

Definition 3:

The form that expresses force $\mathbf{F}$ as the sum of energetic and momentumic forces is called the **momentumic-energetic form**.

Note that the reason this type of force is called the momentumic force in Definition (1) is that, in fact, it can only change the particle's momentum, not its kinetic energy. Another important point concerns the 'generator vector.' Every momentumic force acting on a particle, in order to be perpendicular to the velocity, must be expressed as a vector cross product. In this booklet, the author has named the vector that is cross-multiplied with velocity the 'generator vector.' The term 'generator' is used because this vector, when cross-multiplied with velocity, generates the momentumic force. Each momentumic force has its own corresponding generator vector.

Theorem 1: Suppose that two energetic forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, are simultaneously applied to a point mass that is moving with velocity $\mathbf{v}$. If the mechanical power of the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal, then the vector subtraction of these two forces is a momentumic force.

Proof:

If the difference between the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ is a momentumic force, we can write:

$$\mathbf{F}_2 - \mathbf{F}_1 = \mathbf{f}_m \quad (2)$$

where, $\mathbf{f}_m$ is the momentumic force. If both sides of Equation (2) are dot-multiplied by $\mathbf{v}$, then we have:

$$\mathbf{F}_2 \cdot \mathbf{v} = \mathbf{F}_1 \cdot \mathbf{v} \quad (3)$$

Note that $\mathbf{f}_m \cdot \mathbf{v} = 0$. Equation (3) states that the mechanical power of the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal, and the proof is complete. Note that when $\mathbf{F}_2 = \mathbf{F}_1$, it means that the corresponding generator vector of $\mathbf{f}_m$ is equal to zero.

Theorem 2: Any force applied to a moving point mass can be considered as the resultant of an energetic force and a momentumic force.

Proof:

Suppose that a force $\mathbf{F}$ is exerted to a point mass moving with velocity $\mathbf{v}$, as depicted in Figure (1). We want to express the force $\mathbf{F}$ as the sum of an arbitrary energetic force $\mathbf{F}_E$ and a momentumic force $\mathbf{F}_M$. As shown in Figure (1), it is sufficient to choose the arbitrary force $\mathbf{F}_E$ in such a way that its projection over the velocity $\mathbf{v}$ is equal to the projection of the force $\mathbf{F}$ over the velocity $\mathbf{v}$. Therefore, we have:

$$\mathbf{F}_E \cdot \mathbf{v} = \mathbf{F} \cdot \mathbf{v} \quad (4)$$

Therefore, according to Theorem (1), the momentumic force $\mathbf{F}_M$ can be found such that the following mathematical expression holds:

$$\mathbf{F} = \mathbf{F}_E + \mathbf{F}_M \quad (5)$$

and the proof is complete.

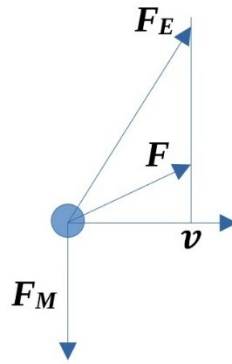


Figure 1. This diagram shows how the force F can be expressed as the sum of two forces F_E and F_M .

3 Existence of Momentomic Force in Nature

3.1 Magnetic Force

When a moving particle with an electric charge q is exposed to a magnetic field $\mathbf{B}$, a force perpendicular to the particle's velocity is exerted on it, known as the magnetic force [6, Ch. 29]. The magnetic force is calculated using the following relation:

$$\mathbf{F} = q \mathbf{v} \times \mathbf{B} \quad (6)$$

Where $\mathbf{v}$ is velocity of the particle. Since the force vector is always perpendicular to the velocity vector in this type of force, the work done is zero. Consequently, the magnetic force can be considered a type of momentomic force.

3.2 Constraint Force

When a particle is constrained to move on a fixed surface or a fixed curve in space, the constraint force acting on it is always perpendicular to the particle's velocity [9, Ch. 2].

3.3 Magnus Force

When a spinning ball has translational motion in a fluid, a force is exerted on it, known as the Magnus force. The Magnus force is calculated using the following formula [10, Ch. 8]:

$$\mathbf{F}_{\text{Magnus}} = C \boldsymbol{\omega} \times \mathbf{v} \quad (7)$$

where $\boldsymbol{\omega}$ is the angular velocity of the ball, $\mathbf{v}$ is the velocity of the ball's centre in a stationary fluid, and C is a constant that depends on the geometry of the ball, the Reynolds number, and the properties of the fluid.

4 Applications of Momentum-Energetic form Force Classification

In the previous sections, the momentum force was defined, and examples of it in nature were provided. In energy analysis of systems, we sometimes encounter equations similar to Equation (3). This is exactly where defining the momentum force can be helpful. If we encounter an equation resembling Equation (3) in the energy analysis of any system, Theorem (1) suggests that a momentum force is certainly involved. Defining the concept of momentum force enables us to express the forces in the momentum-energetic form. This is one of the applications of defining momentum force. Three distinct cases are exemplified in this section.

Case 1: The Momentum-Energetic Form in Eulerian Specification

The Eulerian coordinate system is a coordinate system that is widely used in fluid mechanics. Suppose the velocity vector of a particle can be expressed in terms of its Cartesian position coordinates as $\mathbf{v} = \mathbf{v}(v_x(x, y, z), v_y(x, y, z), v_z(x, y, z))$. Obviously, kinetic energy k can also be expressed as a function of x , y , and z .

-Existence of Momentum Force

If $\mathbf{F}$ is the resultant force acting on the particle, it is obvious that the instantaneous changes in the particle's kinetic energy can be expressed as follows:

$$\dot{k} = \mathbf{F} \cdot \mathbf{v} = \nabla k \cdot \mathbf{v} \quad (8)$$

Equation (8) is similar to Equation (3). Therefore, according to Theorem (1), the force $\mathbf{F}$ can be expressed as the sum of ∇k and a momentum force as follows:

$$\mathbf{F} = \mathbf{f}_m + \nabla k \quad (9)$$

In the next step, the momentum force $\mathbf{f}_m$ is derived.

-Obtaining the Momentum Force

The following vector calculus identity is a key tool for obtaining the momentum force $\mathbf{f}_m$ [11, Ch. 6]:

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v} \cdot \nabla) \mathbf{v} = (\nabla \times \mathbf{v}) \times \mathbf{v} + \frac{1}{2} \nabla (\mathbf{v} \cdot \mathbf{v}) \quad (10)$$

Multiplying both sides of Equation (10) by the mass m gives:

$$\mathbf{F} = m \mathbf{a} = \underbrace{(\nabla \times \mathbf{p}) \times \mathbf{v}}_{\text{momentum force}} + \underbrace{\nabla k}_{\text{energetic force}} \quad (11)$$

Where $\mathbf{p}$ and $\mathbf{a}$ represent the particle's linear momentum and acceleration, respectively. According to Definition (1), the expression $(\nabla \times \mathbf{p}) \times \mathbf{v}$ represents the momentomic force in this case. Note that the following expression is another form for calculating the force $\mathbf{F}$:

$$\mathbf{F} = m v_x \frac{\partial \mathbf{v}}{\partial x} + m v_y \frac{\partial \mathbf{v}}{\partial y} + m v_z \frac{\partial \mathbf{v}}{\partial z} \quad \text{or} \quad \mathbf{F} = m(\mathbf{v} \cdot \nabla) \mathbf{v} \quad (12)$$

Therefore, Equation (11), as an alternative form, expresses the force $\mathbf{F}$ in the momentomic-energetic form.

Case 2: The Momentomic-Energetic Form in Generalized Space: Reformulating Lagrange's Equations of Motion

In the previous case, we observed how, by obtaining the value of $\dot{k}$ in two different ways, we were able to reach an equation similar to Equation (3). In this case, a similar situation will recur in the generalized space, leading to the identification of momentomic forces within the generalized space.

Concept of Momentomic Forces in Generalized Space

In Section (2), the concept of momentomic force was defined in three-dimensional Euclidean vector space. In fact, momentomic force is a force that does not have the ability to do mechanical work, but it can cause a change in the particle's acceleration and momentum. The same concept can be precisely extended to the generalized space.

It is evident that the matrix form of the momentomic force in Equation (1) can be expressed as follows:

$$\mathbf{F} = \begin{pmatrix} 0 & -G_z & G_y \\ G_z & 0 & -G_x \\ -G_y & G_x & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \quad (13)$$

Therefore, the mechanical power of force $\mathbf{F}$ can be expressed as follows:

$$\mathbf{v}^T \begin{pmatrix} 0 & -G_z & G_y \\ G_z & 0 & -G_x \\ -G_y & G_x & 0 \end{pmatrix} \mathbf{v} = 0 \quad (14)$$

Where $\mathbf{v}$ is column matrix of the velocity and $\mathbf{v}^T$ represents transpose of $\mathbf{v}$. Expressing the momentomic force in matrix form in Equation (13) helps us extend this type of force in the generalized space.

-Existence of Momentomic Force

The Jacobi energy function of a mechanical system with n degrees of freedom is defined as follows [12, Ch. 2]:

$$h(q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n; t) = \sum_{j=1}^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \quad (15)$$

Where L is the Lagrangian function; $q_1, \dots, q_n$ are generalized coordinates, and $\dot{q}_1, \dots, \dot{q}_n$ are generalized velocities. Therefore, the derivative of the Jacobi energy function with respect to time is equal to:

$$\dot{h} = \sum_{j=1}^n \ddot{q}_j \frac{\partial L}{\partial \dot{q}_j} + \sum_{j=1}^n \dot{q}_j \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \sum_{j=1}^n \ddot{q}_j \frac{\partial L}{\partial \dot{q}_j} - \sum_{j=1}^n \dot{q}_j \frac{\partial L}{\partial q_j} - \frac{\partial L}{\partial t} \quad (16)$$

From Lagrange's Equations of motion, we have:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j \quad j=1, \dots, n \quad (17)$$

Substituting Equation (17) into Equation (16) gives:

$$\dot{h} = \sum_{j=1}^n Q_j \dot{q}_j - \frac{\partial L}{\partial t} \quad (18)$$

The value of $\dot{h}$ can also be expressed as follows:

$$\dot{h} = \sum_{i=1}^n \frac{\partial h}{\partial \dot{q}_i} \ddot{q}_i + \sum_{i=1}^n \frac{\partial h}{\partial q_i} \dot{q}_i + \frac{\partial h}{\partial t} \quad (19)$$

First, we determine the value of $\frac{\partial h}{\partial \dot{q}_i}$. From Equation (15), the value of h can be expanded as follows:

$$h = \dot{q}_1 \frac{\partial L}{\partial \dot{q}_1} + \dots + \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \dots + \dot{q}_n \frac{\partial L}{\partial \dot{q}_n} - L \quad (20)$$

Taking the partial derivative of both sides of Equation (20) with respect to $\dot{q}_i$ gives:

$$\frac{\partial h}{\partial \dot{q}_i} = \dot{q}_1 \frac{\partial^2 L}{\partial \dot{q}_1 \partial \dot{q}_i} + \dots + \frac{\partial L}{\partial \dot{q}_i} + \dot{q}_i \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_i} + \dots + \dot{q}_n \frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_i} - \frac{\partial L}{\partial \dot{q}_i} = \sum_{j=1}^n \dot{q}_j \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \quad (21)$$

Substituting Equation (21) into Equation (19) gives:

$$\dot{h} = \sum_{i=1}^n \left(\sum_{j=1}^n \dot{q}_j \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \right) \ddot{q}_i + \sum_{i=1}^n \frac{\partial h}{\partial q_i} \dot{q}_i + \frac{\partial h}{\partial t} \quad (22)$$

The expression $\sum_{i=1}^n \left(\sum_{j=1}^n \dot{q}_j \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \right) \ddot{q}_i$ can be rewritten as follows:

$$\begin{aligned} \sum_{i=1}^n \left(\sum_{j=1}^n \dot{q}_j \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \right) \ddot{q}_i &= \left(\dot{q}_1 \frac{\partial^2 L}{\partial \dot{q}_1 \partial \dot{q}_1} \ddot{q}_1 + \dots + \dot{q}_n \frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_1} \ddot{q}_1 \right) + \dots + \left(\dot{q}_1 \frac{\partial^2 L}{\partial \dot{q}_1 \partial \dot{q}_n} \ddot{q}_n + \dots + \dot{q}_n \frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_n} \ddot{q}_n \right) = \\ &= \dot{q}_1 \left(\frac{\partial^2 L}{\partial \dot{q}_1 \partial \dot{q}_1} \ddot{q}_1 + \dots + \frac{\partial^2 L}{\partial \dot{q}_1 \partial \dot{q}_n} \ddot{q}_n \right) + \dots + \dot{q}_n \left(\frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_1} \ddot{q}_1 + \dots + \frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_n} \ddot{q}_n \right) = \sum_{i=1}^n \left(\sum_{j=1}^n \ddot{q}_j \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \right) \dot{q}_i \end{aligned} \quad (23)$$

Substituting Equations (23) and (18) into Equation (22) gives:

$$\sum_{i=1}^n Q_i \dot{q}_i - \frac{\partial L}{\partial t} = \sum_{i=1}^n \left(\sum_{j=1}^n \ddot{q}_j \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \right) \dot{q}_i + \sum_{i=1}^n \frac{\partial h}{\partial q_i} \dot{q}_i + \frac{\partial h}{\partial t} \quad (24)$$

Taking the partial derivative of both sides of Equation (15) with respect to t gives:

$$\frac{\partial h}{\partial t} + \frac{\partial L}{\partial t} = \sum_{i=1}^n \dot{q}_i \frac{\partial^2 L}{\partial \dot{q}_i \partial t} \quad (25)$$

Substituting Equation (25) into Equation (24) gives:

$$\sum_{i=1}^n \left(\sum_{j=1}^n \ddot{q}_j \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} + \frac{\partial h}{\partial q_i} + \frac{\partial^2 L}{\partial \dot{q}_i \partial t} \right) \dot{q}_i = \sum_{i=1}^n Q_i \dot{q}_i \quad (26)$$

Additionally, Equation (26) can be rewritten in matrix form as follows:

$$\dot{\mathbf{q}}^T \begin{pmatrix} \frac{\partial^2 L}{\partial \dot{q}_1 \partial \dot{q}_1} & \dots & \frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 L}{\partial \dot{q}_1 \partial \dot{q}_n} & \dots & \frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_n} \end{pmatrix} \ddot{\mathbf{q}} + \dot{\mathbf{q}}^T \begin{pmatrix} \frac{\partial h}{\partial q_1} + \frac{\partial^2 L}{\partial \dot{q}_1 \partial t} \\ \vdots \\ \frac{\partial h}{\partial q_n} + \frac{\partial^2 L}{\partial \dot{q}_n \partial t} \end{pmatrix} = \dot{\mathbf{q}}^T \mathbf{Q} \quad (27)$$

Where $\ddot{\mathbf{q}} = \begin{pmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_n \end{pmatrix}$, $\dot{\mathbf{q}} = \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix}$.

In Case (1), by writing the instantaneous changes in kinetic energy, we arrived at Equation (8). By comparing Equation (8) and Equation (27), we see that these two equations are very similar to each other. Note that the terms that are multiplied by $\dot{q}_i$ on both sides of Equation (26) have the same dimension, and when they are multiplied by $\dot{q}_i$, they result in the instantaneous changes in energy. The only difference between these two equations is that Equation (8) is written in Euclidean space, while Equation (27) is written in the generalized space.

This is exactly the situation where the concept of momentomic force can be applied. According to Theorem (1), the following expression can be derived from Equation (27):

$$\begin{pmatrix} \frac{\partial^2 L}{\partial \dot{q}_1 \partial \dot{q}_1} & \cdots & \frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 L}{\partial \dot{q}_1 \partial \dot{q}_n} & \cdots & \frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_n} \end{pmatrix} \ddot{\mathbf{q}} + \begin{pmatrix} \frac{\partial h}{\partial q_1} + \frac{\partial^2 L}{\partial \dot{q}_1 \partial t} \\ \vdots \\ \frac{\partial h}{\partial q_n} + \frac{\partial^2 L}{\partial \dot{q}_n \partial t} \end{pmatrix} = \mathbf{Q} + \mathbf{f} \quad (28)$$

Where $\mathbf{f}$ is the generalized momentomic force in the form of a column matrix. Note that according to the definition, the following condition must be satisfied:

$$\dot{\mathbf{q}}^T \mathbf{f} = \sum_{i=1}^n \dot{q}_i f_i = 0 \quad (29)$$

Note that using Theorem (1), the momentomic force cannot be obtained. Theorem (1) only states that if Equation (27) holds, then there is definitely a momentomic force involved in our problem, such that Equation (28) can be concluded. To obtain the value of the momentomic force, other theorems related to the problem must be used. In Case (1), Equation (10), which is a vector identity, was used. In this problem, to obtain the momentomic force, the Lagrange's equations of motion must be used.

-Obtaining the Generalized Momentomic Force

From Lagrange's Equations of Motion, we have:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i \quad i=1, \dots, n \quad (30)$$

Equation (30) can be rewritten as follows:

$$\sum_{j=1}^n \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \ddot{q}_j = - \sum_{j=1}^n \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \dot{q}_j + \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial t} + Q_i \quad i=1, \dots, n \quad (31)$$

Adding $\sum_{j=1}^n \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} \dot{q}_j - \sum_{j=1}^n \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} \dot{q}_j$ to the right side of Equation (31) yields:

$$\sum_{j=1}^n \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \ddot{q}_j = \sum_{j=1}^n \left(\frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \right) \dot{q}_j - \left(\sum_{j=1}^n \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} \dot{q}_j - \frac{\partial L}{\partial q_i} \right) - \frac{\partial^2 L}{\partial \dot{q}_i \partial t} + Q_i \quad i=1, \dots, n \quad (32)$$

Taking the partial derivative of both sides of Equation (15) with respect to q_i gives:

$$\frac{\partial h}{\partial q_i} = \sum_{j=1}^n \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} \dot{q}_j - \frac{\partial L}{\partial q_i} \quad (33)$$

Substituting Equation (33) into Equation (32) gives:

$$\sum_{j=1}^n \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \ddot{q}_j = \sum_{j=1}^n \left(\frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \right) \dot{q}_j - \frac{\partial h}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial t} + Q_i \quad i=1, \dots, n \quad (34)$$

If $\mathbf{P} = \left(\frac{\partial L}{\partial \dot{q}_1} \dots \frac{\partial L}{\partial \dot{q}_n} \right)^T$ is the column matrix of the generalized momentum, then the following expressions hold:

$$\mathbf{J}_P(\mathbf{q}) = \begin{pmatrix} \frac{\partial^2 L}{\partial \dot{q}_1 \partial q_1} & \dots & \frac{\partial^2 L}{\partial \dot{q}_1 \partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 L}{\partial \dot{q}_n \partial q_1} & \dots & \frac{\partial^2 L}{\partial \dot{q}_n \partial q_n} \end{pmatrix} \quad (35)$$

$$\mathbf{J}_P(\dot{\mathbf{q}}) = \begin{pmatrix} \frac{\partial^2 L}{\partial \dot{q}_1 \partial \dot{q}_1} & \dots & \frac{\partial^2 L}{\partial \dot{q}_1 \partial \dot{q}_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 L}{\partial \dot{q}_1 \partial \dot{q}_n} & \dots & \frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_n} \end{pmatrix} \quad (36)$$

Where $\mathbf{J}_P(\mathbf{q})$ and $\mathbf{J}_P(\dot{\mathbf{q}})$ are the Jacobian matrices of $\mathbf{P}$ with respect to $\mathbf{q}$ and $\dot{\mathbf{q}}$, respectively. By substituting $\mathbf{J}_P(\mathbf{q})$ and $\mathbf{J}_P(\dot{\mathbf{q}})$ into Equation (34), Equation (34) can be reformulated in matrix form as follows:

$$\mathbf{J}_P(\dot{\mathbf{q}}) \ddot{\mathbf{q}} = \underbrace{(\mathbf{J}_P^T(\mathbf{q}) - \mathbf{J}_P(\mathbf{q})) \dot{\mathbf{q}}}_{\text{generalized momentum force}} + \underbrace{\mathbf{Q} - \begin{pmatrix} \frac{\partial h}{\partial q_1} + \frac{\partial^2 L}{\partial \dot{q}_1 \partial t} \\ \vdots \\ \frac{\partial h}{\partial q_n} + \frac{\partial^2 L}{\partial \dot{q}_n \partial t} \end{pmatrix}}_{\text{generalized energetic force}} \quad (37)$$

By comparing Equations (37) and (28) with each other, it is clear that the generalized momentum force can be written as follows:

$$\mathbf{f} = (\mathbf{J}_p^T(\mathbf{q}) - \mathbf{J}_p(\mathbf{q}))\dot{\mathbf{q}} \quad (38)$$

The matrix $(\mathbf{J}_p^T(\mathbf{q}) - \mathbf{J}_p(\mathbf{q}))\dot{\mathbf{q}}$ is skew-symmetric. Therefore, we can infer the following expression, which is exactly similar to Equation (14):

$$\dot{\mathbf{q}}^T \mathbf{f} = \dot{\mathbf{q}}^T (\mathbf{J}_p^T(\mathbf{q}) - \mathbf{J}_p(\mathbf{q}))\dot{\mathbf{q}} = 0 \quad (39)$$

Similar to Equation (11), which is in momentum-energetic form, Equation (37) is a reformulated version of Lagrange's equation of motion in momentum-energetic form.

The important point in the second case relates to Equations (26) and (27), which were derived in the part 'Existence of Momentum Force.' When we encounter equations similar to Equation (26) or (27) in the energy analysis of dynamic problems, we can use the definition of momentum force to express the forces in a momentum-energetic form. This approach allows us to solve problems in a more standardized manner. The same concept was examined in Case (1). This is one of the applications of the definition of momentum force mentioned in this booklet. Such a formal definition had not previously existed in scientific literature.

Case 3: The Momentumic-Energetic Form for Conservative Velocity Dependent Forces

Potential energy functions dependent on velocity, which stem from conservative forces, are discussed in various areas of physics, including Lagrangian mechanics [12, Ch. 1]. One of the applications of the definitions presented in this booklet is the ability to define the momentumic-energetic form for velocity-dependent conservative forces, such as the electromagnetic force.

Theorem 3: Consider a conservative force $F(\mathbf{X}, \mathbf{V})$ acting on a particle located at position $\mathbf{X}$ in space and moving with velocity $\mathbf{V}$. Then, the function $F(\mathbf{X}, \mathbf{V})$ is certainly expressed in the momentumic-energetic form as follows:

$$F(\mathbf{X}, \mathbf{V}) = \mathbf{f}(\mathbf{X}) + \mathbf{G}(\mathbf{X}, \mathbf{V}) \times \mathbf{V} \quad (40)$$

Where $\mathbf{G}$ is the generator vector and $\mathbf{f}(\mathbf{X})$ is a vector function such that $\nabla \times \mathbf{f} = \mathbf{0}$.

Proof

If a force is conservative, the work done by it depends only on the initial and final points. This means that we must have:

$$w = \int_A^B \mathbf{f}(\mathbf{X}) \cdot d\mathbf{X} = \int_0^T \mathbf{f}(\mathbf{X}) \cdot \mathbf{V} dt \quad (41)$$

Where $\mathbf{f}(\mathbf{X})$ is a vector function such that $\nabla \times \mathbf{f} = \mathbf{0}$. On the other hand, according to the definition of mechanical work, the work done by the force F is expressed as follows:

$$w = \int_A^B \mathbf{F}(\mathbf{X}, \mathbf{V}) \cdot d\mathbf{X} = \int_0^T \mathbf{F}(\mathbf{X}, \mathbf{V}) \cdot \mathbf{V} dt \quad (42)$$

Since there are no specific conditions for the limits of integration in Equations (41) and (42), we can conclude that the expressions preceding the integrals are equal to each other:

$$\mathbf{F}(\mathbf{X}, \mathbf{V}) \cdot \mathbf{V} = \mathbf{f}(\mathbf{X}) \cdot \mathbf{V} \quad (43)$$

Therefore, according to Theorem (1), the following expression can be concluded:

$$\mathbf{F}(\mathbf{X}, \mathbf{V}) = \mathbf{f}(\mathbf{X}) + \mathbf{G}(\mathbf{X}, \mathbf{V}) \times \mathbf{V} \quad (44)$$

And the proof is complete.

5 Discussion

Comparison of the Conceptualization of Momentum Force and Coriolis Force

The scientific background of what is now called the *Coriolis force* reflects a gradual evolution from qualitative observations to mathematical formulation. Seventeenth-century authors such as Riccioli and Grimaldi (1651) and Dechales (1674) described how the Earth's rotation might alter the paths of falling bodies or projectiles; these remarks were commonly embedded in arguments opposing the motion of the Earth [13]. About a century later, Euler (1749) attempted to express such effects quantitatively [14], [15], and Laplace later incorporated them into his equations of tides and fluid motion [16]. The contribution of Gaspard-Gustave de Coriolis (1835) was to organize these earlier ideas into a systematic dynamical framework for rotating reference systems and to introduce this effect as a new category of force, which he called the *compound centrifugal force* [17]. In short, early writers provided qualitative insights, Euler and Laplace established the mathematical structure, and Coriolis supplied the formal analytical definition that persists today.

In the author's view, this booklet can, in some respects, be considered similar to Coriolis's paper published in 1835. According to historical records, before Coriolis introduced the idea of naming his new force, it had already been recognized in the scientific community, and even its mathematical form had been described by Euler and Laplace. In the case of this booklet, forces that are always perpendicular to velocity exist in nature and are already known in science. What the author of this booklet has done (somewhat similar to Coriolis's work) is to introduce a new name and classification for them.

6 Summary

In this booklet, formal definitions of momentum and energetic forces were presented. Examples of momentum force found in nature were introduced. It was shown that one of the applications of such a new naming becomes apparent when the energy balance equation is written for two different forces, F_1 and F_2 . This means that if the instantaneous powers of the two distinct forces, F_1 and F_2 , are equal, then the third force that connects them is the momentum force introduced in this booklet. This work presents three cases from different situations in which the instantaneous power balance equation for two distinct forces arises, as an application of this new naming. In the second case, it was demonstrated how the Lagrange's equations of motion can be reformulated based on momentum and energetic forces. Furthermore, in Case (3), it was shown that by using the new definitions presented in this booklet, the form of the equation for a velocity-dependent conservative force (such as the electromagnetic force) can be defined.

References

- [1] A. Stinner, "The story of force: from Aristotle to Einstein," *Physics Education*, vol. 29, no. 2, pp. 77–85, 1994.
- [2] M. Jammer, *Concepts of Force*. Mineola, NY: Dover Publications, 1957.
- [3] Aristotle, *On the Heavens*, 301 b 18, W. K. C. Guthrie, Trans. Cambridge, MA: Harvard University Press (Loeb Classical Library), 1939, p. 279.
- [4] G. Galilei, *Discorsi e dimostrazioni matematiche intorno a due nuove scienze (Dialogues concerning two new sciences)*, H. Crew and A. de Salvio, Trans. Evanston, IL: Northwestern Univ. Press, 1946, p. 160.
- [5] I. B. Cohen and G. E. Smith, *The Cambridge Companion to Newton*. Cambridge: Cambridge University Press, 2002, pp. 1–32.
- [6] D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics*. Hoboken, NJ: John Wiley & Sons, 2013.
- [7] A. K. Singh, *Mechanics of Solids*, PHI Learning Pvt. Ltd., 2007, ch. 2.
- [8] A. F. Bower, *Applied Mechanics of Solids*, CRC Press, 2009, ch. 2.
- [9] L. Meirovitch, *Methods of Analytical Dynamics*. Courier Corporation, 2010.
- [10] P. K. Kundu, I. M. Cohen, D. R. Dowling, and J. Capecelatro, *Fluid Mechanics*. Elsevier, 2024.
- [11] P. J. Pritchard and J. W. Mitchell, *Fox and McDonald's Introduction to Fluid Mechanics*. John Wiley & Sons, 2016.
- [12] H. Goldstein, J. Safko, and C. P. Poole, *Classical Mechanics*. Pearson, 2014.
- [13] C. M. Graney, "Coriolis effect, two centuries before Coriolis," *Physics Today*, vol. 64, no. 8, p. 8, 2011, doi: 10.1063/PT.3.1195.
- [14] C. Truesdell, *Essays in the History of Mechanics*. Springer Science & Business Media, 2012, p. 225.
- [15] A. O. Persson, "The Coriolis Effect: Four centuries of conflict between common sense and mathematics, Part I: A history to 1885," *History of Meteorology*, vol. 2, pp. 1–24, 2005.
- [16] D. E. Cartwright, *Tides: A Scientific History*. Cambridge, U.K.: Cambridge University Press, 2000, p. 74.
- [17] G.-G. Coriolis, "Sur les équations du mouvement relatif des systèmes de corps," *Journal de l'École Royale Polytechnique*, vol. 15, pp. 142–154, 1835.